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Simulation of Stochastic Differential Equations Through the Local Linearization Method. A Comparative Study

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A new local linearization (LL) scheme for the numerical integration of non-autonomous multidimensional stochastic differential equations (SDEs) with additive noise is introduced. The numerical scheme is based on the local linearization of the SDE's drift coefficient by means of a truncated Ito–Taylor expansion. A comparative study with the other LL schemes is presented which shows some advantanges of the new scheme over other ones.

KEY WORDS: Local linearization method; stochastic differential equations; numerical solutions.

1. INTRODUCTION

In recent years Stochastic Differential Equations (SDEs) have been increasingly used in modeling complex physical phenomena (see, i.e., ref. 5 and references in ref. 9). Since analytic solutions are rarely available in practical situations, numerical methods to approximate their solutions are required. Up to now a great variety of such numerical methods has been developed (see, i.e., extensive surveys in refs. 5 and 8, and comparative studies by simulations in refs. 7 and 13).

The common theoretical basis of these methods is the stochastic Ito–Taylor expansion of the solution in terms of multiple Wiener integrals.[(5)] In spite of the well known convergence properties achieved by means of this approach, two limitations have been pointed out:[(10, 12)] (1) this approach

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does not give exact solutions for linear ordinary differential equations (i.e., linear SDEs with diffusion coefficient equal to zero), and (2) numerical solutions do not always preserve the qualitative characteristics of the exact solutions. In particular, there are many examples of SDEs with bounded trajectories in which, for any fixed stepsize of the time discretization, the numerical solution becomes explosive when the initial value is in a certain region of the phase space.[10, 2]

An alternative method that attempts to overcome these limitations, called the Local Linearization (LL) method, was introduced at an early stage by Ozaki in 1985.[10, 11] It is based on heuristics directed to obtain an explicit scheme for the numerical solution of the autonomous multidimensional SDE with additive noise

$$dx(t) = f(x)\, dt + dW(t), \quad x(t_0) = x_0, \quad \text{and } W \text{ a Wiener process}$$

in the form of a multivariate autoregressive time series with state-dependent coefficients (it does not involve a stochastic Taylor expansion of the solution of the SDE). But, such formulation of the LL method is ambiguous in regard to the discretization of a random term that appears in the numerical scheme and it does not provide a unique solution for multidimensional SDE.[2] Nevertheless, the original LL scheme gives good results in the simulations of scalar SDEs.

Recently, the LL method was independently reconsidered by Biscay *et al.*[2] and Shoji and Ozaki.[15–17] They introduced new formulations of the LL method oriented to clarify the heuristics of the original one.

The formulation of the LL method by Biscay *et al.*[2] is derived from the following steps: (1) the local linearization of the drift and diffusion coefficients of the SDE in each interval of time $[t, t+h]$ by means of their first order deterministic Taylor expansions, (2) the analytic computation of the solution of the resulting linear SDE, and (3) the approximation of the Ito's integral involved in the solution obtained in the step (2) by means of the simple Trapezoidal rule. This formulation overcomes the previously mentioned shortcomings of the original approach, and allows the LL method to be extended to the case of scalar non-autonomous SDEs with multiplicative noise,

$$dx(t) = f(t, x)\, dt + g(t, x)\, dW(t), \qquad x(t_0) = x_0 \in R$$

and also to the case of multidimensional non-autonomous SDEs with additive noise,

$$dx(t) = f(t, x)\, dt + g(t)\, dW(t), \qquad x(t_0) = x_0 \tag{1}$$

where $x(t) \in R^d$, $g(t) \in R^{d \times m}$, and W is an m-dimensional Wiener process. The explicit scheme derived from this formulation for the equation (1) will be referred as the B-LL scheme. In a comparative study with strong numerical schemes,[2] some advantages of the B-LL scheme are demonstrated. However, it has also shown that the B-LL scheme converges strongly with an order lower than the order of some of the other ones.

The formulation of the LL method by Shoji and Ozaki[15–17] is derived from the following steps: (1) the local linearization of the drift coefficient f in each interval $[t, t+h]$ by applying the Ito formula to f, (2) the analytic computation of the solution of the resulting linear SDE, and (3) the substitution of the Ito's integral involved in the solution obtained in the step (2) by a Gaussian white noise. This was initially proposed for scalar autonomous SDEs with additive noise,[15] and it was generalized later to autonomous multidimensional equations with additive noise.[16, 17] As result of the step (1), the linear approximation derived for f differs of that obtained by the former formulations in a term involving the second derivatives of f. Although this formulation of the LL method also overcomes the shortcomings of the original approach, the authors propose a numerical scheme that is not always computational feasible (it can fail for SDEs for which the Jacobian matrix of the drift coefficient is singular or ill-conditioned in at least a point).

In this paper a new LL scheme for multidimensional non-autonomous SDE with additive noise is introduced. It is proposed as a computational feasible alternative to the Shoji–Ozaki scheme as well as an alternative to the B-LL scheme with greater order of strong convergence. The new scheme essentially combines the approximations that the Shoji–Ozaki formulation and the Biscay *et. al.* formulation provide, respectively, for the drift coefficient f and for the resulting Ito's integral of the step (2) of both formulations.

In Section 2, the new scheme is derived from the following steps: (1) the local linearization of the drift coefficient f in each interval $[t, t+h]$ by means of a truncated Ito–Taylor expansion of f, (2) the analytic computation of the solution of the resulting linear SDE, and (3) the approximation of the Ito's integral involved in the solution obtained in the step (2) by means of the composite Trapezoidal rule. The step (1) here leads to the same approximation to the drift coefficient f as the step (1) of the Shoji–Ozaki formulation but it constitutes a more rigorous theoretical derivation of such approximation. In addition, the similarities and differences between the new LL scheme and the older ones are remarked in this section. In the last section is demonstrated, by means of simulation, that the accuracy and order of convergence of the new scheme are better than those shown by the other computational feasible LL scheme for non-autonomous SDEs: the B-LL scheme.

2. LOCAL LINEARIZATION SCHEMES FOR SDE WITH ADDITIVE NOISE

Consider the multidimensional non-autonomous SDE with additive noise (1). The standard conditions for the existence and uniqueness of a strong solution are assumed. In addition, let the function f be twice continuously differentiable with respect to variable x, and the functions f and g be continuously differentiable with respect to variable t.

Let t be any fixed number in $[t_0, T]$ and $A = \{v, (0), \ldots, (m)\}$ be a hierarchical set,(5, 6) where v denotes the multi-index of length zero. Let

$$f(s, x(s)) = f(t, x(t)) + L^0 f(t, x(t)) \int_t^s du + \sum_{j=1}^{m} L^j f(t, x(t)) \int_t^s dW^j(u) + R_f \tag{2}$$

and

$$x(s) = x(t) + f(t, x(t)) \int_t^s du + g(t) \int_t^s dW(u) + R_x \tag{3}$$

be, respectively, the stochastic Ito–Taylor expansions of function f and x with the hierarchical set A.(5, 6) Here R_f and R_x are the remainder terms,

$$L^j f = (L^j f^1, \ldots, L^j f^d), \qquad \text{for} \quad j = 0, \ldots, m \qquad \text{and} \qquad f = (f^1, \ldots, f^d)$$

$$L^0 f^i = \frac{\partial f^i}{\partial t} + \sum_{k=1}^{d} f^k \frac{\partial f^i}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} g^{k,j} g^{l,j} \frac{\partial^2 f^i}{\partial x^k \partial x^l}, \qquad \text{for} \quad i = 1 \cdots d$$

and

$$L^j f^i = \sum_{k=1}^{d} g^{k,j} \frac{\partial f^i}{\partial x^k}, \qquad \text{for} \quad i = 1 \cdots d \qquad \text{and} \qquad j = 1 \cdots m$$

Combining expansions (2) and (3), and removing the remainder terms, it is found that

$$\begin{aligned} f(s, x(s)) &\approx f(t, x(t)) \\ &\quad + \left(\frac{\partial f(t, x(t))}{\partial t} + \frac{1}{2} \sum_{k,l=1}^{d} [g(t)\, g^{\top}(t)]^{k,l} \frac{\partial^2 f(t, x(t))}{\partial x^k \partial x^l} \right) (s - t) \\ &\quad + J_f(t, x(t))(x(s) - x(t)) \end{aligned} \tag{4}$$

where J_f is the Jacobian matrix of function f and $g^{\top}(t)$ denotes the transpose of matrix $g(t)$.

The linearization of the function f (given by the right side of expression (4)) in the expression (1) leads to the linear SDE

$$dX(s) = (A(t)\, X(s) + a(s, t))\, ds + g(s)\, dW(s) \tag{5}$$

where

$$A(t) = J_f(t, X(t))$$

and

$$\begin{aligned} a(s, t) &= f(t, X(t)) \\ &\quad + \left(\frac{\partial f(t, X(t))}{\partial t} + \frac{1}{2} \sum_{k,l=1}^{d} [g(t)\, g^{\top}(t)]^{k,l}\, \frac{\partial^2 f(t, X(t))}{\partial X^k\, \partial X^l} \right) (s-t) \\ &\quad - J_f(t, X(t))\, X(t) \end{aligned}$$

Let h be a positive number. Since Eq. (5) is an approximation of equation (1) for $s \in [t, t+h]$, its exact solution at the point $s = t+h$ given by[1]

$$\begin{aligned} X(t+h) &= \varphi(t+h) \\ &\quad \times \left(X(t) + \int_t^{t+h} \varphi^{-1}(u)\, a(u, t)\, du + \int_t^{t+h} \varphi^{-1}(u)\, g(u)\, dW(u) \right) \end{aligned} \tag{6}$$

is an approximation to the solution of Eq. (1) at this point. Here $\varphi(u) = \exp(A(t)(u-t))$.

After some algebraic manipulations, expression (6) can be written as

$$X(t+h) = X(t) + \Phi(t, X(t); h) + \xi(t, X(t); h) \tag{7}$$

where

$$\xi(t, X(t); h) = \int_t^{t+h} \varphi(2t + h - u)\, g(u)\, dW(u) \tag{8}$$

is a stochastic process with zero mean and variance

$$\begin{aligned} \Sigma_{\xi}(t, X(t); h) &= E(\xi(t, X(t); h)\, \xi^{\top}(t, X(t); h)) \\ &= \int_t^{t+h} \varphi(2t + h - u)\, g(u)\, g^{\top}(u)\, \varphi^{\top}(2t + h - u)\, du \end{aligned} \tag{9}$$

and

$$\begin{aligned}\Phi(t, X(t); h) = {} & r_0(J_f(t, X(t)), h)\, f(t, X(t)) \\ & + (h r_0(J_f(t, X(t)), h) - r_1(J_f(t, X(t)), h)) \\ & \times \left(\frac{\partial f(t, X(t))}{\partial t} + \frac{1}{2} \sum_{k,\, l=1}^{d} [g(t)\, g^{\top}(t)]^{k,\, l} \frac{\partial^2 f(t, X(t))}{\partial X^k\, \partial X^l} \right) \end{aligned} \tag{10}$$

with

$$r_n(M, a) = \int_0^a \exp(Mu)\, u^n\, du \tag{11}$$

for any positive number a and square matrix M.

Since the integral involved in the definition of ξ is a Ito integral, then

$$\xi(t, X(t); h) = \Psi(t+h)\, W(t+h) - \Psi(t)\, W(t) - \int_t^{t+h} \Psi'(u)\, W(u)\, du$$

where $\Psi(u) = \varphi(2t + h - u)\, g(u)$ and Ψ' denotes the derivative of Ψ with respect to u.[14]

Applying the well-known composite Trapezoidal rule[3] to the integral in the last expression, the following approximation to ξ is obtained

$$\begin{aligned}\tilde{\xi}(t, X(t); h) = {} & \Psi(t+h)\, W(t+h) - \Psi(t)\, W(t) \\ & - \frac{h}{2r} \sum_{k=0}^{r-1} \{\Psi'(\sigma_{k+1})\, W(\sigma_{k+1}) + \Psi'(\sigma_k)\, W(\sigma_k)\} \end{aligned} \tag{12}$$

where $\sigma_k = t + k(h/r)$, and r is an integer number such that $r \sim 1/h$.

Finally, the Local Linearization scheme that approximates the solution of (1) is defined by the iterative computation of the expression

$$X_{t_{n+1}} = X_{t_n} + \Phi(t_n, X_{t_n}; h) + \tilde{\xi}(t_n, X_{t_n}; h) \tag{13}$$

at the discrete times $t_n = t_0 + nh$, $n = 0, 1, 2,...$, starting from $X_{t_0} = x_0$. The integrals r_0 and r_1 in Φ are computing as in ref. 2 by using the Shur method to evaluate matrix functions.

The LL scheme (13) reduces to the B-LL scheme

$$X_{t_{n+1}} = X_{t_n} + r_0(J_f(t_n, X_{t_n}), h)\, f(t_n, X_{t_n})$$
$$+ (hr_0(J_f(t_n, X_{t_n}), h) - r_1(J_f(t_n, X_{t_n}), h))\, \frac{\partial f(t_n, X_{t_n})}{\partial t}$$
$$+ \Psi(t_{n+1})\, W(t_{n+1}) - \Psi(t_n)\, W(t_n)$$
$$- \frac{h}{2} \{\Psi'(t_{n+1})\, W(t_{n+1}) + \Psi'(t_n)\, W(t_n)\} \tag{14}$$

for linear SDEs by setting $r = 1$ in expression (12). Comparing Eqs. (13) and (14) can be see that the differences between the new LL scheme and the B-LL scheme are: (1) an additional term which involve the second derivatives of the drift coefficient, and (2) different Trapezoidal integration rules.

On the other hand, the Shoji–Ozaki scheme for autonomous SDEs is obtained from expression (7) by integrating by part the integrals r_0 and r_1 in Φ and replacing the process ξ by a Gaussian process η with zero mean and variance $\Sigma_\eta(X(s); h) \approx \Sigma_\xi(X(s); h)$ for $s \in [t, t+h)$. Thus, the numerical scheme is defined by the iterative computation of the expression

$$X_{t_{n+1}} = X_{t_n} + J_f^{-1}(X_{t_n})(\exp(hJ_f(X_{t_n})) - I)\, f(X_{t_n})$$
$$+ J_f^{-2}(X_{t_n})(\exp(hJ_f(X_{t_n})) - I - hJ_f(X_{t_n}))\, M(X_{t_n}) + \eta(X_{t_n}; h)$$

at the discrete times $t_n = t_0 + nh$, $n = 0, 1, 2,...$, starting from $X_{t_0} = x_0$. Here, I is the identity matrix, $M(X_{t_n})$ is the trace of the matrix $\frac{1}{2} gg^\top H(X_{t_n})$, and $H(X_{t_n})$ is the Hessian matrix of f at the point X_{t_n}. The variance Σ_η of the process η is defined by the numerical solution of the linear equation

$$J_f(X)\, \Sigma_\eta(X; h) + \Sigma_\eta(X; h)\, J_f^\top(X) = \exp(hJ_f(X))\, gg^\top \exp(hJ_f^\top(X)) - gg^\top$$

which is obtained by integrating by part the integral in (9). However, this numerical scheme is not always computational feasible since it can fail for SDE for which the Jacobian matrix $J_f^{-1}(X)$ is singular or ill-conditioned in at least a point.

It is worth noting that numerical solutions obtained from LL schemes coincide with the exact solutions in the case of linear SDEs with zero noise ($g \equiv 0$), i.e., such schemes are exact for linear ordinary differential equations.

3. NUMERICAL TESTS

The performance of the LL method for the numerical solution of nonlinear SDE has been illustrated in a number of papers (see refs. 10, 11, 12, and 2). A comparative study between the B-LL scheme and the best classical strong schemes has also been carried out,[2] demonstrating some advantages of the B-LL scheme over the other ones. However, it has also demonstrated that the B-LL scheme converges strongly with a global order $\beta = 2$, while some of the other ones converge with a global order $\beta = 3$. In this section is shown, by means of simulations, that the LL scheme (13) introduced in Section 2 converge strongly with a global order $\beta = 3$. We will refer to this scheme as the N-LL scheme.

Denote by $E(\ )$ the mathematical expectation. By definition,[13] a numerical scheme X_{t_n} converges strongly with global order β if

$$E(|x(T) - X_{t_N}|^2 \mid x(t_0) = X_{t_0} = x_0) = O(h^{\beta}) \tag{15}$$

for $t_N = t_0 + Nh = T \gg t_0$ and maximum stepsize $h < 1$. Usually, the left term of the above expression is called global error to distinguishing of the local error $E(|x(t_n) - X_{t_N}|^2 \mid x(t_{n-1}) = X_{t_{n-1}})$. Note that there is another definition of global order β' of strong convergence given by $E(|x(T) - X_{t_N}| \mid x(t_0) = X_{t_0} = x_0) = O(h^{\beta'})$[4] but, it holds that $\beta = 2\beta'$.[5] In this paper we use the definition (15) to maintain the same definition of the previous simulations works.[2, 13]

To take into consideration the effect of the realization of the exact solution by means of pseudo-random numbers generated in a digital computer, the global error is decomposed as in refs. 2 and 13:

$$E(|x(T) - X_{t_N}|^2 \mid x(t_0) = X_{t_0} = x_0)$$
$$\leqslant E(|x(T) - \tilde{X}_{t_N}|^2 \mid x(t_0) = \tilde{X}_{t_0} = x_0) + E(|\tilde{X}_{t_N} - X_{t_N}|^2 \mid \tilde{X}_{t_0} = X_{t_0} = x_0)$$

where $\tilde{X}$ is the discretized exact solution realized by using pseudo-random numbers. $\tilde{X}$ is computed from the expression of the exact solution by replacing the stochastic integrals by the same type of discrete approximation used in the derivation of the numerical scheme, i.e., the approximation (12).

Since in refs. 2 and 13, the quantity $DE = E(|\tilde{X}_{t_N} - X_{t_N}|^2 \mid \tilde{X}_{t_0} = X_{t_0} = x_0)$, called the deterministic part of the error, is used to estimate statistically the order β of the global error. The estimated order $\hat{\beta}$ is obtained from the slope of the straight line fitted to the set of points $\{(\log_2(h_i), \log_2(\widehat{DE}(h_i))\}_{i=1,\ldots,p}$, where $\widehat{DE}(h_i)$ is the estimated of DE corresponding to the stepsize h_i.

For each h, $\widehat{DE}(h)$ is computed as in refs. 5 and 6. For it, the simulations are arranged into M batches of K trajectories of $\widetilde{X}_{t_N}$ each. Let

$$\widehat{DE}_{i,j}(h) = |\widetilde{X}^{i,j}_{t_N} - X^{i,j}_{t_o+Nh}|^2$$

be the square error at the end of the interval $[t_o, T]$ for the jth trajectory of the ith batch and let

$$\widehat{DE}_i(h) = \frac{1}{K} \sum_{j=1}^{K} \widehat{DE}_{i,j}(h), \qquad \text{and} \qquad \widehat{DE}(h) = \frac{1}{M} \sum_{i=1}^{M} \widehat{DE}_i(h)$$

be the sample means of the ith batch and of all batches, respectively. From the Student's t-distribution with $M-1$ degrees of freedom, a $100(1-\alpha)\%$ confidence interval for $DE(h)$ has the form

$$(\widehat{DE}(h) - \Delta, \widehat{DE}(h) + \Delta)$$

where

$$\Delta = t_{1-\alpha/2, M-1} \sqrt{\frac{\hat{\sigma}^2_{DE}}{M}}, \qquad \hat{\sigma}^2_{DE} = \frac{1}{M-1} \sum_{i=1}^{M} |\widehat{DE}_i(h) - \widehat{DE}(h)|^2$$

$t_{1-\alpha/2, M-1}$ is determined from the Student's t-distribution with $M-1$ degree of freedom and $0 < \alpha < 1$.

In the following examples the global order β of convergence of the B-LL and N-LL schemes are estimated. For each example the estimated value of DE and its 90% ($\alpha = 0.1$) confidence interval are computed for the stepsizes $h_i = 2^{-(i+4)}$ with $i = 1,..., 4$. The simulations are arranged into $M = 20$ batches of $N = 100$ trajectories each to obtain small confidence intervals.

Example 1. An scalar autonomous SDE (Example 4.53 in ref. 5).

Let

$$dx(t) = (\exp(-x(t)) + 1)\, dt + \tfrac{5}{2}\, dW(t), \qquad t \in [0, 1], \quad x(0) = \tfrac{1}{2}$$

be a scalar autonomous SDE, and

$$x(t) = \frac{1}{2} + t + \frac{5}{2} W(t) + \ln\left(1 + \int_0^t \exp\left(-\frac{1}{2} - u - \frac{5}{2} W(u)\right) du\right)$$

its exact solution.

Table I. Estimated Values of *DE* and Their Respective 90% Confidence Intervals Obtained for Each Scheme in Example 1

B-LL scheme			N-LL scheme		
h	$\widehat{DE}(h)$	$\pm\Delta$	h	$\widehat{DE}(h)$	$\pm\Delta$
2^{-5}	0.00024535	±0.00002299	2^{-5}	0.0000260404	±0.0000045884
2^{-6}	0.00006396	±0.00000470	2^{-6}	0.0000035891	±0.0000004423
2^{-7}	0.00001620	±0.00000102	2^{-7}	0.0000005356	±0.0000000953
2^{-8}	0.00000376	±0.00000023	2^{-8}	0.0000000582	±0.0000000050

Table I presents the estimated values of DE and their respective 90% ($\alpha = 0.1$) confidence intervals for each scheme. Figure 1 shows the straight line fitted to the points $\{(\log_2(h_i), \log_2(\widehat{DE}(h_i))\}_{i=1,...,4}$ corresponding to each numerical scheme. The estimated slope of these lines (with 90% confidence interval) are $\hat{\beta} = 2.00 \pm 0.08$ for the B-LL scheme and $\hat{\beta} = 2.91 \pm 0.20$ for the N-LL scheme. Note that the straight line corresponding to the N-LL scheme is completely under the other one, what means that the N-LL scheme shows better accuracy.

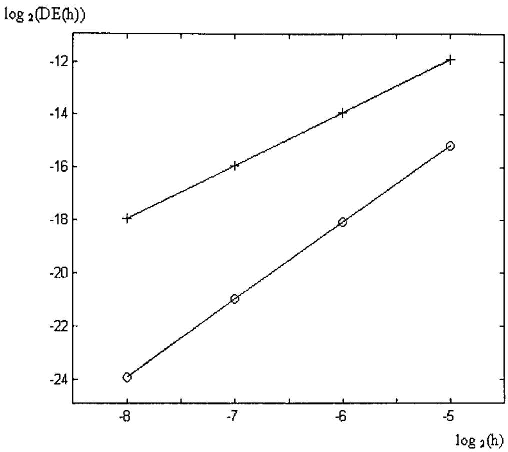

Fig. 1. Estimated values of DE obtained from the B-LL scheme ($+$) and the N-LL scheme ($\circ$) solutions of the Example 1 corresponding to the stepsizes $h_i = 2^{-(i+4)}$ with $i = 1,..., 4$. The estimated slope from least-squares line fitting of the points $(\log_2(h_i), \log_2(\widehat{DE}(h_i))$ are, respectively, $\hat{\beta} = 2.00 \pm 0.08$ and $\hat{\beta} = 2.91 \pm 0.20$ with a 90% confidence intervals. The coefficient of correlation between the variables $\log_2(h)$ and $\log_2(\widehat{DE}(h))$ is $R > 0.9989$ for each numerical scheme.

Example 2. A two-dimensional autonomous SDE

Let

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} dt + G \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}, \qquad t \in [0, 1], \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

be a two-dimensional autonomous SDE, where

$$f_1(x_1, x_2) = \exp(-2x_1(t) - x_2(t)) - \exp(-x_1(t) - x_2(t))$$
$$f_2(x_1, x_2) = 2\exp(-x_1(t) - x_2(t)) - \exp(-2x_1(t) - x_2(t)) + 1$$

and

$$G = \frac{5}{2} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Its exact solution is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z_2(t, x_2(t_0)/2) - z_1(t, x_1(t_0) + x_2(t_0)/2) \\ 2z_1(t, x_1(t_0) + x_2(t_0)/2) - z_2(t, x_2(t_0)/2) \end{bmatrix}$$

where

$$z_i(t, C) = C + t + \frac{5}{2} W_i(t) + \ln\left(1 + \int_0^t \exp\left(-C - u - \frac{5}{2} W_i(u)\right) du\right)$$

Table II presents the estimated values of DE for the two schemes. Figure 2 shows, for each numerical scheme, the straight line fitted to the points $\{(\log_2(h_i), \log_2(\widehat{DE}(h_i))\}_{i=1,\dots,4}$ for each variable. The estimated slopes of these lines are presented in Table III. These results show that the accuracy and order of convergence of the N-LL scheme are significantly better for both variables.

Example 3. A two-dimensional non-autonomous SDE (Example 3 in ref. 2).

Let

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2) \end{bmatrix} dt + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} dW(t), \;\; t \in [2, 3], \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 50 \end{bmatrix}$$

Table II. Estimated Values of *DE* and Their Respective 90% Confidence Intervals Obtained for the Variables X_1 and X_2 of Example 2 for Each Numerical Scheme

B-LL scheme			N-LL scheme		
h	$\widehat{DE}(h)$	$\pm\Delta$	h	$\widehat{DE}(h)$	$\pm\Delta$
2^{-5}	0.000011147	±0.000001154	2^{-5}	0.0000011949	±0.0000001854
2^{-6}	0.000002838	±0.000000151	2^{-6}	0.0000001637	±0.0000000193
2^{-7}	0.000000672	±0.000000031	2^{-7}	0.0000000209	±0.0000000016
2^{-8}	0.000000163	±0.000000007	2^{-8}	0.0000000025	±0.0000000002
2^{-5}	0.00025113	±0.00002572	2^{-5}	0.000023931	±0.000004516
2^{-6}	0.00006085	±0.00000404	2^{-6}	0.000003322	±0.000000426
2^{-7}	0.00001421	±0.00000098	2^{-7}	0.000000510	±0.000000100
2^{-8}	0.00000370	±0.00000026	2^{-8}	0.000000054	±0.000000005

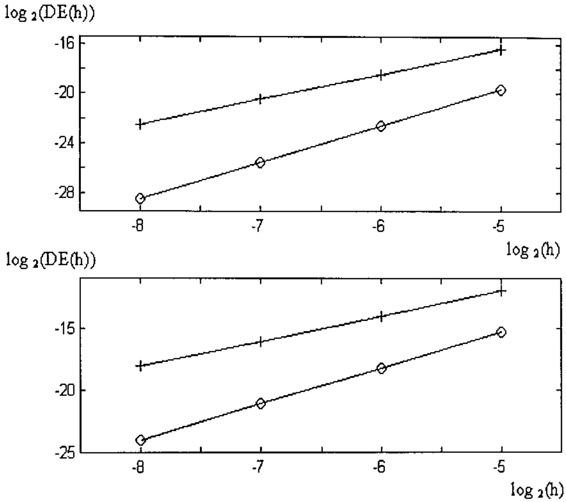

Fig. 2. Estimated values of *DE* obtained from the B-LL scheme (+) and the N-LL scheme (∘) solutions of the Example 2 corresponding to the stepsizes $h_i = 2^{-(i+4)}$ with $i = 1,\ldots, 4$. (a) for variable x_1 and (b) for variable x_2. See in Table III the estimated slope $\hat{\beta}$ from least-squares line fitting of the points $(\log_2(h_i), \log_2(\widehat{DE}(h_i))$ for each variable and numerical scheme. The coefficient of correlation between the variables $\log_2(h)$ and $\log_2(\widehat{DE}(h))$ is $R > 0.9986$ for each variable and numerical scheme.

Table III. Estimated Slopes $\hat{\beta}$ from Least-Squares Line Fitting of the Stepsize-Error Points of Fig. 2 Corresponding to Example 2 (with 90% Confidence Intervals)

Variable\scheme	B-LL	N-LL
x_1	2.03 ± 0.04	2.96 ± 0.08
x_2	2.03 ± 0.06	2.90 ± 0.22

be a two-dimensional non-autonomous SDE, where

$$f_1(t, x_1, x_2) = C \exp\left(-\frac{x_1(t)}{C}\right)$$

$$f_2(t, x_1, x_2) = \frac{C}{t^2} \exp\left(-\frac{x_1(t)}{C}\right) + 4\left(x_2(t) - \frac{x_1(t)}{t^2}\right)\frac{\cos(t)}{(2+\sin(t))} - 2\,\frac{x_1(t)}{t^3}$$

$$g_1(t) = \sqrt{2}\,C$$

$$g_2(t) = \sqrt{2}\,C\left(\frac{1}{t^2} + \frac{D(2+\sin(t))^4}{(t+1)}\right)$$

Its exact solution is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,C\left(W(t) + \frac{1}{\sqrt{2}}\ln\left|\exp\left(\frac{x_1(t_0)}{C}\right) + \int_{t_0}^{t} \exp(-\sqrt{2}\,W(u))\,du\right|\right) \\ \frac{x_1(t)}{t^2} + D(2+\sin(t))^4\left(x_2(t_0) - \frac{x_1(t_0)}{t_0^2} + \sqrt{2}\,C\int_{t_0}^{t}\frac{dW(u)}{u+1}\right) \end{bmatrix}$$

where $C = 30/\sqrt{2}$ and $D = (2+\sin(t_0))^{-4}$.

As in the previous examples, estimated values of DE and β are computed for both schemes. Tables IV and V present, respectively these estimates. Figure 3 shows, for each numerical scheme, the straight line fitted to the points $\{(\log_2(h_i), \log_2(\widehat{DE}(h_i))\}_{i=1,\dots,4}$ for each variable. It is observed that the accuracy and order of convergence of the N-LL scheme are significantly better for both variables.

Table IV. Estimated Values of *DE* and Their Respective 90% Confidence Intervals Obtained for the Variables X_1 and X_2 of Example 3 for Each Numerical Scheme

B-LL scheme			N-LL scheme		
h	$\widehat{DE}(h)$	$\pm\Delta$	h	$\widehat{DE}(h)$	$\pm\Delta$
2^{-5}	0.016643	±0.000786	2^{-5}	0.00137588	±0.00014601
2^{-6}	0.003972	±0.000153	2^{-6}	0.00018284	±0.00001459
2^{-7}	0.000960	±0.000033	2^{-7}	0.00002385	±0.00000183
2^{-8}	0.000242	±0.000008	2^{-8}	0.00000290	±0.00000016
2^{-5}	0.00054572	±0.00002554	2^{-5}	0.0000548114	±0.0000060337
2^{-6}	0.00014046	±0.00000468	2^{-6}	0.0000075729	±0.0000006281
2^{-7}	0.00002923	±0.00000091	2^{-7}	0.0000007748	±0.0000000616
2^{-8}	0.00000676	±0.00000022	2^{-8}	0.0000000841	±0.0000000049

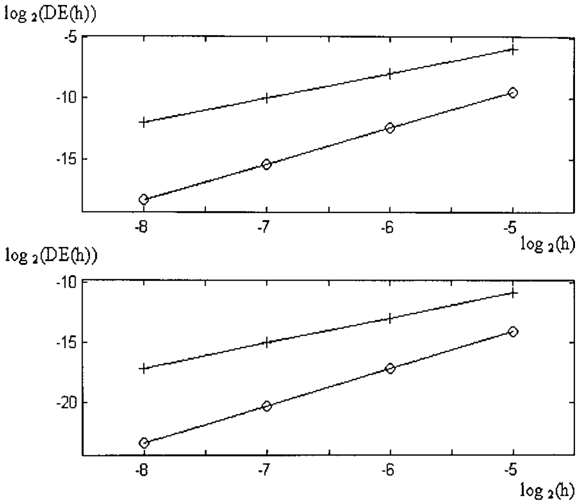

Fig. 3. Estimated values of *DE* obtained from the B-LL scheme (+) and the N-LL scheme (∘) solutions of the Example 3 corresponding to the stepsizes $h_i = 2^{-(i+4)}$ with $i = 1,..., 4$. (a) for variable x_1 and (b) for variable x_2. See in Table V the estimated slope $\hat{\beta}$ from least-squares line fitting of the points $(\log_2(h_i), \log_2(\widehat{DE}(h_i))$ for each variable and numerical scheme. The coefficient of correlation between the variables $\log_2(h)$ and $\log_2(\widehat{DE}(h))$ is $R > 0.9991$ for each variable and numerical scheme.

Table V. Estimated Slopes $\hat{\beta}$ from Least-Squares Line Fitting of the Stepsize-Error Points of Fig. 3 Corresponding to Example 3 (with 90% Confidence Intervals)

Variable\scheme	B-LL	N-LL
x_1	2.03 ± 0.04	2.96 ± 0.05
x_2	2.12 ± 0.12	3.13 ± 0.19

4. CONCLUSIONS

A new Local Linearization scheme for the numerical integration of Stochastic Differential Equations (SDEs) with additive noise was introduced. The numerical scheme is based on the local linearization of the SDE's drift coefficient by means of a truncated Ito–Taylor expansion. Some advantages of the new scheme over the other LL schemes were pointed out. The simulation study carried out demonstrates that new scheme has better accuracy and order of strong convergence.

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